

OPERATOR DIAGONALIZATIONS OF MULTIPLIER SEQUENCES

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ABSTRACT. We consider hyperbolicity preserving operators with respect to a new linear operator representation on $\mathbb{R}[x]$. In essence, we demonstrate that every Hermite and Laguerre multiplier sequence can be diagonalized into a sum of hyperbolicity preserving operators, where each of the summands forms a classical multiplier sequence. Interestingly, this does not work for other orthogonal bases; for example, this property fails for the Legendre basis. We establish many new formulas concerning the Q_k 's of Peetre's 1959 differential representation for linear operators in the specific case of Hermite and Laguerre diagonal differential operators. Additionally, we provide a new algebraic characterization of the Hermite multiplier sequences and also extend a recent result of T. Forgács and A. Piotrowski on hyperbolicity properties of the polynomial coefficients in hyperbolicity preserving Hermite diagonal differential operators.

1. INTRODUCTION

Define the Jacobi-Theta function by,

$$\Phi(t) := \sum_{n=1}^{\infty} (2n^4 \pi^2 e^{9t} - 3n^2 \pi e^{5t}) e^{-n^2 \pi e^{4t}}. \quad (1)$$

It is well known that the Riemann Hypothesis [34, (1859)] is equivalent to the statement that the integral cosine transform of the Jacobi-Theta function,

$$\int \Phi(t) \cos(xt) dt, \quad (2)$$

can be uniformly approximable by polynomials with only real zeros (see for example G. Csordas, T. Norfolk, and R. Varga [15]) (see also [13, 14, 16, 17]). In 1913, J. Jensen [22] showed that every entire function,

$$f(x) := \sum_{k=0}^{\infty} \frac{\gamma_k}{k!} x^k, \quad (3)$$

can be uniformly approximated by polynomials with only real zeros if and only if $g_n(x)$ has only real zeros for each $n \in \mathbb{N}_0$, where

$$g_n(x) := \sum_{k=0}^n \binom{n}{k} \gamma_k x^k. \quad (4)$$

Hence, the associated Jensen polynomials, $\{g_n(x)\}_{n=0}^{\infty}$, have received a great deal of attention in modern times (see for example [9–11, 18]). In particular, M. Chasse in 2011 showed remarkably that the first $2 \cdot 10^{17}$ Jensen polynomials of (2) have only real zeros [8, Theorem 177, p. 87].

In 1914, G. Pólya and J. Schur [32] gave a complete characterization of hyperbolicity preserving operators (operators that map polynomials with only real zeros to those of the same kind, see Definition 4) of the form,

$$T[x^n] := \gamma_n x^n, \quad \{\gamma_n\}_{n=0}^{\infty} \subset \mathbb{R}, \quad (5)$$

by showing that $f(x)$ (from (3)) must be uniformly approximable by polynomials with zeros of one sign (throughout the literature, $\{\gamma_n\}_{n=0}^{\infty}$ is called a multiplier sequence). Their work was greatly extended in 2009 by J. Borcea and P. Brändén [5] who demonstrated that essentially every linear operator written in J. Peetre's [28, (1959)] differential operator form,

$$T := \sum_{k=0}^{\infty} Q_k(x) D^k, \quad D := \frac{d}{dx}, \quad (6)$$

is hyperbolicity preserving if and only if either

$$T[e^{xw}] := e^{xw} \sum_{k=0}^{\infty} Q_k(x) w^k \quad \text{or} \quad T[e^{-xw}] := e^{-xw} \sum_{k=0}^{\infty} Q_k(x) (-w)^k \quad (7)$$

is uniformly approximable by real two-variable stable polynomials.

It was shown by Laguerre in 1882 [23] (later generalized by G. Pólya [30, (1913)], [31, (1915)]) that every real entire function, which can be uniformly approximated by polynomials with only real zeros, must be of the form,

$$f(x) := cx^m e^{-ax^2+bx} \prod_{k=1}^{\omega} \left(1 + \frac{x}{x_k}\right) e^{-x/x_k}, \quad (8)$$

where $0 \leq \omega \leq \infty$, $a, b, c \in \mathbb{R}$, $a \geq 0$, $m \in \mathbb{N}_0$, $\{x_k\}_{k=1}^{\omega} \subset \mathbb{R}$, $x_k \neq 0$, and $\sum_{k=1}^{\omega} \frac{1}{x_k^2} < \infty$. Likewise, real entire functions, that can be uniformly approximated by polynomials with non-positive zeros, must be of the form,

$$f(x) := cx^m e^{bx} \prod_{k=1}^{\omega} \left(1 + \frac{x}{x_k}\right), \quad (9)$$

where $0 \leq \omega \leq \infty$, $b, c \in \mathbb{R}$, $b \geq 0$, $m \in \mathbb{N}_0$, $\{x_k\}_{k=1}^{\omega}$, $x_k > 0$, and $\sum_{k=1}^{\omega} \frac{1}{x_k} < \infty$. In 1983, T. Craven and G. Csordas demonstrated an important subclass of functions from (9), showing that $b \geq 1$ if and only if $f^{(k)}(0) \leq f^{(k+1)}(0)$ ($\gamma_k \leq \gamma_{k+1}$) for every $k \in \mathbb{N}_0$ [10]. This motivated re-investigating some of P. Turán's results [37, (1954)] by modifying G. Pólya and J. Schur's operator, equation (5), replacing x^n with the n^{th} Hermite polynomial (see [4]). In 2007, A. Piotrowski gave a complete characterization of hyperbolicity preserving operators that diagonalize on the Hermite basis,

$$T[H_n(x)] := \gamma_n H_n(x), \quad \{\gamma_n\}_{n=0}^{\infty} \subset \mathbb{R}, \quad (10)$$

where for each $n \in \mathbb{N}_0$, $H_n(x)$ denotes the n^{th} Hermite polynomial. It was demonstrated that T is hyperbolicity preserving if and only if $\{\gamma_n\}_{n=0}^{\infty}$ is an increasing classical multiplier sequence (from (5)) (see Theorem 11). Recently, there has been significant motivation in characterizing multiplier sequences of any basis (see [1–3, 8, 19–21, 29, 38]). In particular, it has become increasingly apparent, the role that orthogonal polynomials seem to play in defining hyperbolicity preserving operators (see also the recent characterization of Laguerre multiplier sequences by P. Brändén and E. Ottergren [6], see Theorem 12).

In this paper, we modify J. Peetre's differential representation [28], giving a new differential representation for study with respect to hyperbolicity preservation (Theorem 17 and 19). We use this to essentially show that every Hermite and Laguerre multiplier sequence can be written as a sum of classical multiplier sequences (Theorem 32 and 45). Interestingly, the Legendre basis does not enjoy this property (Example 25). New methods of determining the differential representation of Hermite and Laguerre diagonal differential operators are found (Theorem 36, 38, and 47). Additionally, we give a new algebraic characterization of Hermite multiplier sequences (Theorem 41) and generalize a recent statement of T. Forgács and A. Piotrowski [20], on the hyperbolicity properties of the Q_k 's in (6) that arise from a Hermite diagonal differential operator (Theorem 40).

Definition 1. We will denote the *Hermite*, *Laguerre*, and *Legendre* polynomials as, $\{H_n(x)\}_{n=0}^{\infty}$, $\{L_n(x)\}_{n=0}^{\infty}$, and $\{P_n(x)\}_{n=0}^{\infty}$, respectively [33, pp. 157, 187, 201]. For each $n \in \mathbb{N}_0$, these polynomials are given by the following formulas,

$$H_n(x) = \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{(-1)^k n! 2^{n-2k}}{k!(n-2k)!} x^{n-2k}, \quad (11)$$

$$L_n(x) = \sum_{k=0}^n \frac{(-1)^k}{k!} \binom{n}{k} x^k, \quad \text{and} \quad (12)$$

$$P_n(x) = \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{(-1)^k}{2^n} \binom{n}{k} \binom{2n-2k}{n} x^{n-2k}. \quad (13)$$

It is well known that these polynomials satisfy the following differential equations [33, pp. 173, 188, 204, 258],

$$((-1/2)D^2 + (x)D) H_n(x) = (n)H_n(x), \quad (14)$$

$$((-x)D^2 + (x-1)D) L_n(x) = (n)L_n(x), \quad \text{and} \quad (15)$$

$$((x^2-1)D^2 + (2x)D) P_n(x) = (n^2+n)P_n(x), \quad (16)$$

where $D := \frac{d}{dx}$.

Definition 2. Suppose $f(x)$ is an entire function,

$$f(x) := \sum_{k=0}^{\infty} \frac{\gamma_k}{k!} x^k. \quad (17)$$

For each $n \in \mathbb{N}_0$, we define the n^{th} *Jensen polynomial* associated to the entire function $f(x)$ (or associated to the sequence $\{\gamma_k\}_{k=0}^{\infty}$) by,

$$g_n(x) := \sum_{k=0}^n \binom{n}{k} \gamma_k x^k. \quad (18)$$

Likewise, for each $n \in \mathbb{N}_0$, we define the n^{th} *reversed Jensen polynomial* by,

$$g_n^*(x) := \sum_{k=0}^n \binom{n}{k} \gamma_k x^{n-k}. \quad (19)$$

Definition 3. Let $T : \mathbb{R}[x] \rightarrow \mathbb{R}[x]$ be a linear operator such that $T[B_n(x)] = \gamma_n B_n(x)$ for every $n \in \mathbb{N}_0$, where $\{\gamma_n\}_{n=0}^{\infty}$ is a sequence of real numbers and $\{B_n(x)\}_{n=0}^{\infty}$, $\deg(B_n(x)) = n$, $B_0 \neq 0$, is a basis of real polynomials. Then T will be referred to as a *diagonal differential operator* with respect to the eigenvector sequence, $\{B_n(x)\}_{n=0}^{\infty}$, and eigenvalue sequence, $\{\gamma_n\}_{n=0}^{\infty}$. If $\{B_n(x)\}_{n=0}^{\infty} = \{x^n\}_{n=0}^{\infty}$ then T is said to be a *classical diagonal differential operator*. Similarly, if $\{B_n(x)\}_{n=0}^{\infty}$ is the Hermite, Laguerre, or Legendre polynomials (Definition 1), then T is said to be a *Hermite diagonal differential operator*, *Laguerre diagonal differential operator*, or a *Legendre diagonal differential operator*, respectively.

Definition 4. Let $T : \mathbb{R}[x] \rightarrow \mathbb{R}[x]$ be a linear operator. Operator T is said to be *hyperbolicity preserving* if $T[p(x)]$ has only real zeros whenever $p(x) \in \mathbb{R}[x]$ has only real zeros. If in addition, T diagonalizes on $\{B_n(x)\}_{n=0}^{\infty} = \{x^n\}_{n=0}^{\infty}$, $\{B_n(x)\}_{n=0}^{\infty} = \{H_n(x)\}_{n=0}^{\infty}$, $\{B_n(x)\}_{n=0}^{\infty} = \{L_n(x)\}_{n=0}^{\infty}$, or $\{B_n(x)\}_{n=0}^{\infty} = \{P_n(x)\}_{n=0}^{\infty}$, as in $T[B_n(x)] = \gamma_n B_n(x)$ for some sequence of real numbers, $\{\gamma_n\}_{n=0}^{\infty}$, then $\{\gamma_n\}_{n=0}^{\infty}$ is called a *classical multiplier sequence*, *Hermite multiplier sequence*, *Laguerre multiplier sequence*, or *Legendre multiplier sequence*, respectively.

Definition 5. Suppose T is a hyperbolicity preserving operator that diagonalizes on $\{B_n(x)\}_{n=0}^{\infty}$ and $\{\gamma_n\}_{n=0}^{\infty}$, where

$$\{\gamma_n\}_{n=0}^{\infty} := \{0, 0, \dots, 0, 0, \alpha, \beta, 0, 0, 0, \dots\}, \quad \alpha, \beta \in \mathbb{R}. \quad (20)$$

Then, $\{\gamma_n\}_{n=0}^{\infty}$ is called a *trivial multiplier sequence*. In Theorems 11 and 12 we will exclude all trivial multiplier sequences.

Definition 6. The *Laguerre-Pólya class*, denoted as $\mathcal{L}-\mathcal{P}$, is the set of entire functions that are uniform limits of *hyperbolic polynomials*, real valued polynomials with only real zeros. We define $\mathcal{L}-\mathcal{P}^s$ to be the entire functions in $\mathcal{L}-\mathcal{P}$ with Taylor coefficients of the same sign. Likewise, we define $\mathcal{L}-\mathcal{P}^a$ to be the entire functions in $\mathcal{L}-\mathcal{P}$ with alternating Taylor coefficients. The notation, $\mathcal{L}-\mathcal{P}^{sa}$, is defined as $\mathcal{L}-\mathcal{P}^{sa} := \mathcal{L}-\mathcal{P}^s \cup \mathcal{L}-\mathcal{P}^a$. Given an interval, $I \subseteq \mathbb{R}$, $\mathcal{L}-\mathcal{P}^* I$ will denote functions in $\mathcal{L}-\mathcal{P}^*$ that have zeros only in I , where $\mathcal{L}-\mathcal{P}^*$ is either $\mathcal{L}-\mathcal{P}$, $\mathcal{L}-\mathcal{P}^s$, $\mathcal{L}-\mathcal{P}^a$, or $\mathcal{L}-\mathcal{P}^{sa}$.

Theorem 7 (T. Craven and G. Csordas [10, (1983)]). Suppose $f(x) \in \mathcal{L}-\mathcal{P}^s$. Then $|f^{(k)}(0)| \leq |f^{(k+1)}(0)|$ for all $k \in \mathbb{N}_0$ if and only if $e^{-x} f(x) \in \mathcal{L}-\mathcal{P}^s$.

Remark 8. In the sequel we will make use of the fact that many of the classes defined above are closed under differentiation. Consider an entire function,

$$f(x) := \sum_{k=0}^{\infty} \frac{\gamma_k}{k!} x^k. \quad (21)$$

If $f(x)$ is in $\mathcal{L}-\mathcal{P}^s$, $\mathcal{L}-\mathcal{P}^a$, or $\mathcal{L}-\mathcal{P}$, then for every $n \in \mathbb{N}_0$, $f^{(n)}(x)$ is also in $\mathcal{L}-\mathcal{P}^s$, $\mathcal{L}-\mathcal{P}^a$, or $\mathcal{L}-\mathcal{P}$, respectively. Similarly, a slight extension of Theorem 7 shows that if $e^{-\sigma x} f(x) \in \mathcal{L}-\mathcal{P}^s$ ($\sigma > 0$), then for every $n \in \mathbb{N}_0$, $e^{-\sigma x} f^{(n)}(x) \in \mathcal{L}-\mathcal{P}^s$. Likewise, if $e^{\sigma x} f(x) \in \mathcal{L}-\mathcal{P}^a$ ($\sigma > 0$), then for every $n \in \mathbb{N}_0$, $e^{\sigma x} f^{(n)}(x) \in \mathcal{L}-\mathcal{P}^a$.

Theorem 9 ([1], [28], [29, Proposition 29, p. 32]). If $T : \mathbb{R}[x] \rightarrow \mathbb{R}[x]$ is any linear operator, then there is a unique sequence of real polynomials, $\{Q_k(x)\}_{k=0}^{\infty} \subset \mathbb{R}[x]$, such that

$$T = \sum_{k=0}^{\infty} Q_k(x) D^k, \quad \text{where } D := \frac{d}{dx}. \quad (22)$$

Furthermore, given any sequence of polynomials, $\{B_n(x)\}_{n=0}^\infty$ ($\deg(B_n(x)) = n$ for each $n \in \mathbb{N}_0$, $B_0(x) \neq 0$), then for each $n \in \mathbb{N}_0$,

$$Q_n(x) = \frac{1}{B_n^{(n)}} \left(T[B_n(x)] - \sum_{k=0}^{n-1} Q_k(x) B_n^{(k)}(x) \right). \quad (23)$$

Theorem 10 (G. Pólya and J. Schur [32, (1914)]). *Let $\{\gamma_k\}_{k=0}^\infty$ be a sequence of real numbers. Sequence $\{\gamma_k\}_{k=0}^\infty$ is a positive or negative multiplier sequence if and only if*

$$\sum_{k=0}^\infty \frac{\gamma_k}{k!} x^k \in \mathcal{L}\text{-}\mathcal{P}^s. \quad (24)$$

Sequence $\{\gamma_k\}_{k=0}^\infty$ is an alternating multiplier sequence if and only if

$$\sum_{k=0}^\infty \frac{\gamma_k}{k!} x^k \in \mathcal{L}\text{-}\mathcal{P}^a. \quad (25)$$

Theorem 11 (A. Piotrowski [29, Theorem 152, p. 140 (2007)]). *Let $\{\gamma_k\}_{k=0}^\infty$ be a sequence of real numbers and let $\{g_k^*(x)\}_{k=0}^\infty$ be the sequence of reversed Jensen polynomials associated with $\{\gamma_k\}_{k=0}^\infty$. Sequence $\{\gamma_k\}_{k=0}^\infty$ is a non-trivial positive or negative Hermite multiplier sequence if and only if*

$$e^{-x} \sum_{k=0}^\infty \frac{\gamma_k}{k!} x^k = \sum_{k=0}^\infty \frac{g_k^*(-1)}{k!} x^k \in \mathcal{L}\text{-}\mathcal{P}^s. \quad (26)$$

Sequence $\{\gamma_k\}_{k=0}^\infty$ is a non-trivial alternating Hermite multiplier sequence if and only if

$$e^x \sum_{k=0}^\infty \frac{\gamma_k}{k!} x^k = e^{2x} \sum_{k=0}^\infty \frac{g_k^*(-1)}{k!} x^k \in \mathcal{L}\text{-}\mathcal{P}^a. \quad (27)$$

Theorem 12 (P. Brändén and E. Ottergren [6, (2014)]). *Let $\{\gamma_k\}_{k=0}^\infty$ be a sequence of real numbers and let $\{g_k^*(x)\}_{k=0}^\infty$ be the reversed Jensen polynomials associated with $\{\gamma_k\}_{k=0}^\infty$. Sequence $\{\gamma_k\}_{k=0}^\infty$ is a non-trivial positive or negative Laguerre multiplier sequence if and only if*

$$\sum_{k=0}^\infty g_k^*(-1) x^k \in \mathbb{R}[x] \cap \mathcal{L}\text{-}\mathcal{P}^s[-1, 0]. \quad (28)$$

There are no non-trivial alternating Laguerre multiplier sequences.

From Theorem 10, 11, and 12, it is clear that every Laguerre multiplier sequence is a Hermite multiplier sequence, and every Hermite multiplier sequence is a classical multiplier sequence (see also the classification diagram of K. Blakeman, E. Davis, T. Forgács, and K. Urabe [3]).

In the literature, it is common to discuss only the non-negative multiplier sequences. However, many of our results establish strong differences between the sequences from $\mathcal{L}\text{-}\mathcal{P}^a$ and the sequences in $\mathcal{L}\text{-}\mathcal{P}^s$ (see for example Theorem 39). Hence, we will take great care to discuss $\mathcal{L}\text{-}\mathcal{P}^s$ sequences separately from $\mathcal{L}\text{-}\mathcal{P}^a$ sequences. The following example demonstrates the strong differences in differential representation from positive eigenvalues versus alternating eigenvalues.

Example 13. Consider the following hyperbolicity preserving Hermite diagonal differential operators (see Theorem 11),

$$T[H_n(x)] := nH_n(x) \quad \text{and} \quad W[H_n(x)] := (-1)^n nH_n(x). \quad (29)$$

Using the recursive formula from Theorem 9, we calculate T and W ,

$$T = (x)D + \left(-\frac{1}{2}\right)D^2, \quad (30)$$

and

$$W = (-x)D + \left(2x^2 - \frac{1}{2}\right)D^2 + (-2x^3 + x)D^3 + \cdots. \quad (31)$$

We observe that T is a finite order differential operator, while W is an infinite order differential operator. This observation makes sense when we note that $\{(-1)^n n\}_{n=0}^\infty$ is not interpolatable by a polynomial (see [1]).

The sensitivity of the two classes, $\mathcal{L}\text{-}\mathcal{P}^s$ and $\mathcal{L}\text{-}\mathcal{P}^a$, can also be seen in the following theorem, which holds for sequences arising from $\mathcal{L}\text{-}\mathcal{P}^s$, but not for sequences arising from $\mathcal{L}\text{-}\mathcal{P}^a$.

Theorem 14 (T. Craven and G. Csordas [10, (1983)]). *Let $\{\gamma_k\}_{k=0}^\infty$ be a positive or negative classical multiplier sequence. Then, for each $m \in \mathbb{N}_0$,*

$$\left\{ \sum_{k=0}^n \binom{n}{k} \gamma_{m+k} \right\}_{n=0}^\infty \quad \text{and} \quad \left\{ \sum_{k=0}^m \binom{m}{k} \gamma_{n+k} \right\}_{n=0}^\infty, \quad (32)$$

are also positive or negative classical multiplier sequences, respectively.

Proof. For the first sequence, using a Cauchy product, we calculate

$$\sum_{n=0}^\infty \left(\sum_{k=0}^n \binom{n}{k} \gamma_{m+k} \right) \frac{x^n}{n!} = e^x D^m \sum_{n=0}^\infty \frac{\gamma_n}{n!} x^n \in \mathcal{L}\text{-}\mathcal{P}^s. \quad (33)$$

For the second sequence, using two Cauchy products, we calculate

$$\sum_{n=0}^\infty \left(\sum_{k=0}^m \binom{m}{k} \gamma_{n+k} \right) \frac{x^n}{n!} = e^{-x} D^m e^x \sum_{n=0}^\infty \frac{\gamma_n}{n!} x^n \in \mathcal{L}\text{-}\mathcal{P}^s. \quad \square$$

\square

Example 15. We show that Theorem 14 does not hold for $\mathcal{L}\text{-}\mathcal{P}^a$. Consider the following function in $\mathcal{L}\text{-}\mathcal{P}^a$,

$$f(x) := \sum_{k=0}^\infty \frac{(-1)^k}{k!} \frac{x^k}{k!}, \quad (34)$$

which is obtained by application of the multiplier sequence $\left\{ \frac{(-1)^k}{k!} \right\}_{k=0}^\infty$ to the function e^x . The sequence,

$$\{\gamma_n\}_{n=0}^\infty = \left\{ \sum_{k=0}^n \binom{n}{k} \frac{(-1)^k}{k!} \right\}_{n=0}^\infty, \quad (35)$$

has the form,

$$\{\gamma_n\}_{n=0}^\infty = \left\{ 1, 0, -\frac{1}{2}, -\frac{2}{3}, -\frac{5}{8}, \dots, \frac{887}{5760}, \dots \right\}. \quad (36)$$

Hence, $\{\gamma_n\}_{n=0}^\infty$ is not a multiplier sequence, since there is no function in $\mathcal{L}\text{-}\mathcal{P}^s$ or $\mathcal{L}\text{-}\mathcal{P}^a$ with Taylor coefficients that match the signs of $\{\gamma_n\}_{n=0}^\infty$ (see Theorem 10).

For the reader's convenience we provide the following compilation of combinatorial identities that will be used extensively throughout the paper. These types of calculations have already been observed in the proof of Theorem 14.

Theorem 16 ([35, p. 49], [29, Proposition 33, p. 35]). *Given a sequence of real numbers, $\{\alpha_k\}_{k=0}^\infty$, for each $n \in \mathbb{N}_0$, define,*

$$\beta_n = \sum_{k=0}^n \binom{n}{k} \alpha_k. \quad (37)$$

Then, for all $n \in \mathbb{N}_0$,

$$\alpha_n = \sum_{k=0}^n \binom{n}{k} \beta_k (-1)^{n-k}. \quad (38)$$

In particular, we have,

$$e^x \sum_{n=0}^\infty \frac{\alpha_n}{n!} x^n = \sum_{k=0}^\infty \frac{\beta_k}{k!} x^k \quad \text{and} \quad e^{-x} \sum_{k=0}^\infty \frac{\beta_k}{k!} x^k = \sum_{n=0}^\infty \frac{\alpha_n}{n!} x^n. \quad (39)$$

Similarly, if $\{g_k^(x)\}_{k=0}^\infty$ are the reversed Jensen polynomials associated with $\{\gamma_k\}_{k=0}^\infty$, then for every $n \in \mathbb{N}_0$,*

$$\gamma_n = \sum_{k=0}^n \binom{n}{k} g_k^*(-1) \quad \text{and} \quad g_n^*(-1) = \sum_{k=0}^n \binom{n}{k} \gamma_k (-1)^{n-k}. \quad (40)$$

Likewise, if $\{\gamma_k\}_{k=0}^\infty$ diagonalizes the classical diagonal differential operator, T , then

$$T[x^n] = \left(\sum_{k=0}^\infty \frac{g_k^*(-1)}{k!} x^k D^k \right) x^n = \gamma_n x^n. \quad (41)$$

2. OPERATOR DIAGONALIZATIONS OF DIAGONALIZABLE OPERATORS

Our main objective is to present a new representation of diagonal differential operators (Theorem 17). We will only need to assume that $\deg(Q_k(x)) \leq k$ for each $k \in \mathbb{N}_0$; a property that all diagonal differential operators have, as the recursive formula of Theorem 9 shows (see also [1]).

Theorem 17. *Given a linear operator, $T : \mathbb{R}[x] \rightarrow \mathbb{R}[x]$,*

$$T = \sum_{k=0}^{\infty} Q_k(x) D^k, \quad (42)$$

where $\deg(Q_k(x)) \leq k$ for every $k \in \mathbb{N}_0$. Define the family of sequences,

$$\{b_{n,k}\}_{k=0}^{\infty} := \left\{ \sum_{j=0}^k \binom{k}{j} Q_{j+n}^{(j)}(0) \right\}_{k=0}^{\infty}, \quad n \in \mathbb{N}_0. \quad (43)$$

For each $n \in \mathbb{N}_0$, define the classical diagonal differential operator,

$$T_n[x^k] := b_{n,k} x^k. \quad (44)$$

Then,

$$T = \sum_{n=0}^{\infty} T_n D^n. \quad (45)$$

Furthermore, the representation in (45) is unique.

Proof. We are concerning ourselves with operators defined on $\mathbb{R}[x]$, hence convergence discussions are a non-issue. By Theorem 16, for every $n \in \mathbb{N}_0$, we know the differential representation of T_n , namely,

$$T_n = \sum_{k=0}^{\infty} \left(\sum_{j=0}^k \binom{k}{j} b_{n,j} (-1)^{k-j} \right) \frac{1}{k!} x^k D^k = \sum_{k=0}^{\infty} \frac{Q_{k+n}^{(k)}(0)}{k!} x^k D^k. \quad (46)$$

Note the calculation $\frac{Q_{k+n}^{(k)}(0)}{k!} x^k$ is precisely the k^{th} term of the polynomial, $Q_{k+n}(x)$. Hence, each summand, in each T_n , is one term from some $Q_k(x)$. Furthermore, no two T_n 's use the same term in a particular $Q_k(x)$. Finally, because $\deg(Q_k(x)) \leq k$, we are assured that every term in every $Q_k(x)$ will be present in some T_n . The uniqueness follows from the uniqueness of the differential representation in Theorem 9. \square

Example 18. Theorem 17 can be best understood with the aid of a concrete illustrative. Define the differential operator,

$$T := \underbrace{(a_2 x^2 + b_1 x + c_0)}_{Q_2(x)} D^2 + \underbrace{(a_1 x + b_0)}_{Q_1(x)} D + \underbrace{(a_0)}_{Q_0(x)}, \quad (47)$$

where $a_2, a_1, a_0, b_1, b_0, c_0 \in \mathbb{R}$. Using Theorem 17, we re-write T , in terms of T_n 's,

$$\begin{aligned} T &= \left(\frac{Q_2^{(2)}(0)}{2!} x^2 D^2 + \frac{Q_1^{(1)}(0)}{1!} x^1 D^1 + \frac{Q_0^{(0)}(0)}{0!} x^0 D^0 \right) D^0 + \\ &\quad \left(\frac{Q_2^{(1)}(0)}{1!} x^1 D^1 + \frac{Q_1^{(0)}(0)}{0!} x^0 D^0 \right) D^1 + \\ &\quad \left(\frac{Q_2^{(0)}(0)}{0!} x^0 D^0 \right) D^2 \\ &= \underbrace{(a_2 x^2 D^2 + a_1 x D + a_0)}_{T_0} + \underbrace{(b_1 x D + b_0)}_{T_1} D + \underbrace{(c_0)}_{T_2} D^2. \end{aligned} \quad (48)$$

Theorem 17 can be extended to arbitrary linear operators on $\mathbb{R}[x]$; reminiscent of a Laurent series from complex variables (see [25, p. 222]).

Theorem 19. *Let $T : \mathbb{R}[x] \rightarrow \mathbb{R}[x]$ be an arbitrary linear operator,*

$$T := \sum_{k=0}^{\infty} Q_k(x) D^k. \quad (49)$$

Define the family of sequences,

$$\{b_{n,k}\}_{k=0}^{\infty} := \left\{ \sum_{j=0}^k \binom{k}{j} Q_{j+n}^{(j)}(0) \right\}_{k=0}^{\infty}, \quad n \in \mathbb{Z}, \quad (50)$$

where we take $Q_{j+n}^{(j)}(0) = 0$ for $n + j < 0$. For each $n \in \mathbb{Z}$, define the classical diagonal differential operator,

$$T_n[x^k] := b_{n,k}x^k. \quad (51)$$

Then,

$$T = \sum_{n=1}^{\infty} T_{-n}D^{-n} + \sum_{n=0}^{\infty} T_nD^n, \quad (52)$$

where we define $D \cdot D^{-1} = 1$. Furthermore, the representation in (52) is unique.

Proof. We first note that for each $n \in \mathbb{N}_0$, $T_n = \sum_{k=0}^{\infty} \frac{Q_{k+n}^{(k)}(0)}{k!} x^k D^k$ (see Theorem 16). Similar to the proof of Theorem 17, each term from the T_n 's are in one-to-one correspondence with each term in the Q_k 's. Thus, a change of index yields,

$$T = \sum_{n=0}^{\infty} Q_n(x)D^n = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{\infty} \frac{Q_k^{(k)}(0)}{k!} x^k \right) D^n = \sum_{n=-\infty}^{\infty} \left(\sum_{k=0}^{\infty} \frac{Q_{k+n}^{(k)}(0)}{k!} x^k \right) D^{k+n} = \sum_{n=-\infty}^{\infty} T_n D^n. \quad \square$$

Example 20. We provide another example demonstrating Theorem 19. Define the differential operator,

$$T := (a_2x^2 + b_1x + c_0)D^2 + (z_1x^2 + a_1x + b_0)D + (y_0x^2 + z_0x + a_0), \quad (53)$$

where $y_0, z_1, z_0, a_2, a_1, a_0, b_1, b_0, c_0 \in \mathbb{R}$. Using Theorem 19, we rewrite T in terms of T_n 's,

$$T = \underbrace{(y_0x^2D^2)}_{T_{-2}} D^{-2} + \quad (54)$$

$$\underbrace{(z_1x^2D^2 + z_0xD)}_{T_{-1}} D^{-1} + \quad (55)$$

$$\underbrace{(a_2x^2D^2 + a_1xD + a_0)}_{T_0} D^0 + \quad (56)$$

$$\underbrace{(b_1xD + b_0)}_{T_1} D^1 + \quad (57)$$

$$\underbrace{(c_0)}_{T_2} D^2. \quad (58)$$

Example 21. It is possible for representation (52) to be “transcendental” in both directions. Consider the differential operator,

$$T := \sum_{k=0}^{\infty} (x^{2k} + 1)D^k. \quad (59)$$

Then for $n \in \mathbb{N}$, $T_{-n} = x^{2n}D^{2n}$ and for $n \in \mathbb{N}_0$, $T_n = 1$. Hence,

$$T = \cdots + T_{-2}D^{-2} + T_{-1}D^{-1} + T_0D^0 + T_1D^1 + T_2D^2 + \cdots \quad (60)$$

$$= \cdots + (x^4D^4)D^{-2} + (x^2D^2)D^{-1} + (1)D^0 + (1)D^1 + (1)D^2 + \cdots. \quad (61)$$

Upon attaining the representation (45) in Theorem 17, we direct our attention to the property of hyperbolicity preservation. If T in equation (45), is hyperbolicity preserving, then what properties do the T_n 's possess? One might hope that the T_n 's also enjoy the property of hyperbolicity preservation. This hope would certainly be warranted since, in fact, T_0 always possess the property of hyperbolicity preservation in a diagonal differential operator (see [1] and [29, Theorem 158, p. 145]). In addition, classical multiplier sequences and operators of the form $f(xD)$ and $f(D)$, from the Hermite-Poulain [27, p. 4] and Laguerre Theorems [27, Satz 3.2], trivially have T_n 's that are hyperbolicity preserving. However, in general, our hope is false as the next several examples will demonstrate. The following Turán type inequality, equation (63), will be of great use.

Theorem 22 (R. Bates and R. Yoshida [2, (2013)]). *Let $a, b, c, r_1, r_2, r_3 \in \mathbb{R}$. Define polynomials $Q_2(x) = a(x - r_1)(x - r_2)$, $Q_1(x) = b(x - r_3)$, and $Q_0(x) = c$. Then T is hyperbolicity preserving, where*

$$T := Q_2(x)D^2 + Q_1(x)D + Q_0(x), \quad (62)$$

if and only if a, b, c are of the same sign and

$$b^2 \left(\frac{(r_1 - r_3)(r_3 - r_2)}{(r_1 - r_2)^2} \right) - ac \geq 0. \quad (63)$$

We take $\left(\frac{(r_1 - r_3)(r_3 - r_2)}{(r_1 - r_2)^2} \right) = \frac{1}{4}$ when $r_1 = r_2 = r_3$. If $r_1 = r_2$ and $r_1 \neq r_3$, then T is not hyperbolicity preserving.

Remark 23. For clarity, we point out that the condition that a, b, c be of the same sign, in Theorem 22, cannot be removed. For example, the following operator satisfies equation (63) but not the necessary sign condition of the leading coefficients,

$$T := (x - 1)(x + 1)D^2 - 2xD + 1. \quad (64)$$

Hence, T is not hyperbolicity preserving, as can be seen since $T[x^2] = -x^2 - 2$.

Example 24. Consider the following differential operator,

$$T := (x - 2)(x + 1)D^2 + 3(x + 1/2)D + 1 \quad (65)$$

$$= (-2)D^2 + (-xD + 3/2)D + (x^2D^2 + 3xD + 1) \quad (66)$$

$$= T_2D^2 + T_1D + T_0. \quad (67)$$

By an application of Theorem 22, operator T is certainly hyperbolicity preserving,

$$3^2 \left(\frac{(2 - (-1/2))((-1/2) - (-1))}{((-1) - 2)^2} \right) - 1 \cdot 1 = \frac{1}{4} \geq 0. \quad (68)$$

However, $T_1 = -xD + 3/2$ (see (44)) is not a hyperbolicity preserver, since $T_1[x^2 - 1] = (-1/2)x^2 - 3/2$.

Example 25. Consider the Legendre basis of polynomials, $\{P_n(x)\}_{n=0}^\infty$, that satisfy the differential equation (Definition 1),

$$((x^2 - 1)D^2 + (2x)D + 1)P_n(x) = (n^2 + n + 1)P_n(x). \quad (69)$$

Equation (69) was first verified to be hyperbolicity preserving by K. Blakeman, E. Davis, T. Forgács, and K. Urabe [3, Lemma 5]. We re-verify that $(x^2 - 1)D^2 + (2x)D + 1$ is a hyperbolicity preserver using the calculation in Theorem 22,

$$2^2 \left(\frac{(1 - 0)(0 - (-1))}{(-1 - 1)^2} \right) - 1 \cdot 1 = 1 - 1 = 0 \geq 0. \quad (70)$$

Hence, compositions are hyperbolicity preserving, and thus, T is hyperbolicity preserving, where $T[P_n(x)] := (n^2 + n + 1)^3 P_n(x)$. We calculate the differential form of T (see Theorem 9),

$$T = ((x^2 - 1)D^2 + (2x)D + 1)^3 \quad (71)$$

$$= (x^6 - 3x^4 + 3x^2 - 1)D^6 + \quad (72)$$

$$(18x^5 - 36x^3 + 18x)D^5 + \quad (73)$$

$$(101x^4 - 130x^2 + 29)D^4 + \quad (74)$$

$$(208x^3 - 160x)D^3 + \quad (75)$$

$$(145x^2 - 57)D^2 + \quad (76)$$

$$(26x)D + \quad (77)$$

$$1. \quad (78)$$

Consider the highlighted terms of from above to calculate T_4 (see (44)),

$$T_4 = 3x^2D^2 + 18xD + 29. \quad (79)$$

From Theorem 22 we infer that operator T_4 fails to be hyperbolicity preserving,

$$18^2 \left(\frac{1}{4} \right) - 3 \cdot 29 = 81 - 87 = -6 < 0. \quad (80)$$

Example 26. Due to A. Piotrowski (see [29, Lemma 157, p. 145]), affine transforms $\{c_n B_n(\alpha x + \beta)\}_{n=0}^\infty$, $c_n, \alpha, \beta \in \mathbb{R}$, $c_n, \alpha \neq 0$) share the same multiplier sequence class as the basis $\{B_n(x)\}_{n=0}^\infty$. Let us consider then an affine transform of the Hermite polynomials, $\{H_n(x \pm 3)\}_{n=0}^\infty$, and a multiplier sequence for these shifted Hermite polynomials, $\{n^2 + n + 1\}_{n=0}^\infty$ (see Theorem 11). Thus T is hyperbolicity preserving, where $T[H_n(x \pm 3)] = (n^2 + n + 1)H_n(x \pm 3)$. We calculate the differential form of T (see Theorem 9),

$$T = \left(\frac{1}{4}\right) D^4 + (-x \mp 3) D^3 + \left(x^2 \pm 6x + \frac{15}{2}\right) D^2 + (2x \pm 6) D + (1). \quad (81)$$

From the highlighted items in (81) we formulate $T_2 = -xD + 15/2$ (see (44)) and note that T_2 is not hyperbolicity preserving since $T[2x^8 - 2x^6] = -x^8 - 3x^6$.

It is intriguing to see that while affine transforms share multiplier sequence classes, the T_n 's in equation (45) may not share in the property of hyperbolicity preservation. Hence, as we will see in Theorem 31 and 32, the Hermite polynomials are distinguished amongst all affine transforms of the Hermite polynomials.

Example 27. Consider the shifted Laguerre polynomials (see [29, Lemma 157, p. 145]), $\{L_n(x+2)\}_{n=0}^\infty$, and a multiplier sequence for these shifted Laguerre polynomials, $\{n\}_{n=0}^\infty$ (see Theorem 12). Thus T is hyperbolicity preserving, where $T[L_n(x+2)] = nL_n(x+2)$ and

$$T = (-x - 2)D^2 + (x + 1)D + (0). \quad (82)$$

Consider the operator formed by the highlighted terms, $T_1 = -xD + 1$. Operator T_1 fails to preserve hyperbolicity since $T_1[x^2 - 1] = -x^2 - 1$. (See also Question 2 in the open problems.)

Example 28. A more technical example is the following. Using the generalized Malo-Schur-Szegő Composition Theorem [7, 12] it can be shown that, given $p(x) = (x+1)^3$,

$$T := -\frac{1}{6}p'''(x)D^3 + \frac{1}{2}p''(x)D^2 - p'(x)D + p(x) \quad (83)$$

$$= -D^3 + (3x + 3)D^2 + (-3x^2 - 6x - 3)D + (x^3 + 3x^2 + 3x + 1) \quad (84)$$

is hyperbolicity preserving [38, p. 47]. Define $T_1 := 3xD - 3$ (see (44)) and note that $T_1[x^2 - 1] = 3x^2 + 3$, thus T_1 is not hyperbolicity preserving.

Example 29. Another example involving Q_k 's, where $\deg(Q_k(x)) > k$ for some of the k 's. Using the Hermite-Poulain Theorem [27, p. 4] it can be shown that the non-diagonalizable operator,

$$T := (x^2 + 2x + 1)D^2 - (x^2 + 2x + 1), \quad (85)$$

preserves hyperbolicity. The operator $T_0 = x^2D^2 - 1$ (see (44)) is not a hyperbolicity preserver, since $T_0[x^2 - 1] = x^2 + 1$. This example is even more interesting considering the fact that, in general, W_0 is always hyperbolicity preserving, whenever W is any arbitrary diagonal differential hyperbolicity preserver (see [1]).

By now the reader has hopefully been convinced that Examples 24-29 demonstrate the very high sensitivity of the following results; namely, for Hermite or Laguerre multiplier sequences the T_n 's in (44) from Theorem 17 are hyperbolicity preservers. It is surprising, that not only will each T_n be hyperbolicity preserving, the family of sequences, $\{b_{n,k}\}_{k=0}^\infty$ (see (43)), turn out to be more Hermite or Laguerre multiplier sequences, respectively. In this sense every Hermite or Laguerre multiplier sequence generates an entire family of additional Hermite or Laguerre multiplier sequences.

3. OPERATOR DIAGONALIZATIONS OF HERMITE MULTIPLIER SEQUENCES

Our main goal in this section is to demonstrate for hyperbolicity preserving Hermite diagonal differential operators, each T_n defined in Theorem 17 is hyperbolicity preserving. This will be done in two phases. First we will find a formula for $b_{n,k}$ (see (43)). Second, we will show that, for each $n \in \mathbb{N}_0$, $\{b_{n,k}\}_{k=0}^\infty$ a Hermite multiplier sequence and hence $\{b_{n,k}\}_{k=0}^\infty$ is also a classical multiplier sequence, i.e. each T_n is hyperbolicity preserving.

Lemma 30. For $k, j \in \mathbb{N}_0$, the k^{th} derivative of the $(k+2j+1)^{th}$ and $(k+2j)^{th}$ Hermite polynomials (see Definition 1) evaluated at zero is,

$$H_{k+2j+1}^{(k)}(0) = 0 \quad \text{and} \quad H_{k+2j}^{(k)}(0) = \frac{(k+2j)!2^k(-1)^j}{j!}. \quad (86)$$

Theorem 31. Let T be a Hermite diagonal differential operator, $T[H_n(x)] := \gamma_n H_n(x)$, where $\{\gamma_n\}_{n=0}^\infty$ a sequence of real numbers. Then there is a sequence of polynomials, $\{Q_k(x)\}_{k=0}^\infty$, and a sequence of classical diagonal differential operators, $\{T_n\}_{n=0}^\infty$, such that

$$T[H_n(x)] := \left(\sum_{k=0}^\infty Q_k(x) D^k \right) H_n(x) = \left(\sum_{k=0}^\infty T_k D^k \right) H_n(x) = \gamma_n H_n(x).$$

Then, for each $n \in \mathbb{N}_0$,

$$\{b_{2n+1,m}\}_{m=0}^\infty = \{0\}_{m=0}^\infty$$

and

$$\{b_{2n,m}\}_{m=0}^\infty := \left\{ \sum_{k=0}^m \binom{m}{k} \frac{(-1)^n}{n! 2^n} \left(\sum_{j=0}^n \binom{n}{j} \frac{g_{k+n+j}^*(-1)}{2^j} \right) \right\}_{m=0}^\infty,$$

where $T_n[x^m] = b_{n,m} x^m$ for every $n, m \in \mathbb{N}_0$.

Proof. The exists of the sequences $\{Q_k(x)\}_{k=0}^\infty$ and $\{T_k\}_{k=0}^\infty$ are established by Theorem 9 and 17. We now begin with the remarkable representation formula of T. Forgács and A. Piotrowski that computes the Q_k 's in any Hermite diagonal differential operator [20, Theorem 3.1],

$$Q_k(x) = \sum_{j=0}^{\lfloor k/2 \rfloor} \frac{(-1)^j}{j!(k-2j)! 2^{k-j}} g_{k-j}^*(-1) H_{k-2j}(x). \quad (87)$$

This formula yields the following expressions for all $k, n \in \mathbb{N}_0$,

$$Q_{k+2n+1}^{(k)}(0) = 0, \quad (88)$$

and

$$Q_{k+2n}^{(k)}(0) = \frac{(-1)^n}{n! 2^n} \sum_{j=0}^n \binom{n}{j} \frac{g_{k+n+j}^*(-1)}{2^j}. \quad (89)$$

Equations (88) and (89) could have been calculated using the recursive formula of Theorem 9, if one knew, *a priori*, the importance of the $g_{k-j}^*(-1)$'s in formula (87). However, this dependence was not made apparent until formula (87) was uncovered.

Let us now verify (88) and (89). Equation (88) is obvious from formula (87) and the fact that the Hermite polynomials alternate between even and odd polynomials. We now establish (89) using formula (87) and Lemma 30 as follows:

$$Q_{k+2n}^{(k)}(0) = \sum_{j=0}^{\lfloor (k+2n)/2 \rfloor} \frac{(-1)^j}{j!(k+2n-2j)! 2^{k+2n-j}} g_{k+2n-j}^*(-1) H_{k+2n-2j}^{(k)}(0) \quad (90)$$

$$= \sum_{j=0}^n \frac{(-1)^j}{j!(k+2(n-j))! 2^{k+n+(n-j)}} g_{k+n+(n-j)}^*(-1) H_{k+2(n-j)}^{(k)}(0) \quad (91)$$

$$= \sum_{j=0}^n \frac{(-1)^{n-j}}{(n-j)!(k+2j)! 2^{k+n+j}} g_{k+n+j}^*(-1) H_{k+2j}^{(k)}(0) \quad (92)$$

$$= \sum_{j=0}^n \frac{(-1)^{n-j}}{(n-j)!(k+2j)! 2^{k+n+j}} g_{k+n+j}^*(-1) \left(\frac{(k+2j)! 2^k (-1)^j}{j!} \right) \quad (93)$$

$$= \frac{(-1)^n}{n! 2^n} \sum_{j=0}^n \binom{n}{j} \frac{g_{k+n+j}^*(-1)}{2^j}. \quad (94)$$

We finish the proof by using formula (43). \square

With the aid of what has been shown thus far, we are now in a position to demonstrate our main result, that every Hermite multiplier sequence is the unique sum of classical multiplier sequences. That is, for Hermite multiplier sequences, each T_n in equation (44) is hyperbolicity preserving. The spirit of the following argument will be the establishment of a Rodrigues type formula that relates each governing entire function, $\sum_{k=0}^\infty \frac{b_{n,k}}{k!} x^k$, of each T_n , with the entire function that defines the hyperbolicity properties of T itself, $\sum_{k=0}^\infty \frac{\gamma_k}{k!} x^k$ (see Theorem 10 and 11).

Theorem 32. Let $\{\gamma_k\}_{k=0}^\infty$ is a non-trivial Hermite multiplier sequence and let $\{g_k^*(x)\}_{k=0}^\infty$ be the reversed Jensen polynomials associated with $\{\gamma_k\}_{k=0}^\infty$. Then, for each $n \in \mathbb{N}_0$,

$$\{b_{n,m}\}_{m=0}^\infty := \left\{ \sum_{k=0}^m \binom{m}{k} \frac{(-1)^n}{n!2^n} \left(\sum_{j=0}^n \binom{n}{j} \frac{g_{k+n+j}^*(-1)}{2^j} \right) \right\}_{m=0}^\infty, \quad (95)$$

is a Hermite multiplier sequence.

Proof. By assumption, $\{\gamma_k\}_{k=0}^\infty$ is a Hermite multiplier sequence. Hence, by Theorem 11, if

$$f(x) := \sum_{k=0}^\infty \frac{f^{(k)}(0)}{k!} x^k := \sum_{k=0}^\infty \frac{g_k^*(-1)}{k!} x^k, \quad (96)$$

then, either $f(x) \in \mathcal{L}-\mathcal{P}^s$ or $e^{2x}f(x) \in \mathcal{L}-\mathcal{P}^a$. We wish to show that, $\{b_{n,m}\}_{m=0}^\infty$, is a Hermite multiplier sequence; thus using Theorem 11 we must show that if

$$h_n(x) := \sum_{m=0}^\infty \left(\sum_{k=0}^m \binom{m}{k} b_{n,k} (-1)^{m-k} \right) \frac{x^m}{m!}, \quad (97)$$

then either $h_n(x) \in \mathcal{L}-\mathcal{P}^s$ or $e^{2x}h_n(x) \in \mathcal{L}-\mathcal{P}^a$. We use Theorem 16 and perform the following calculation,

$$h_n(x) = \sum_{m=0}^\infty \left(\sum_{k=0}^m \binom{m}{k} b_{n,k} (-1)^{m-k} \right) \frac{x^m}{m!} \quad (98)$$

$$= \sum_{k=0}^\infty \left(\frac{(-1)^n}{n!2^n} \left(\sum_{j=0}^n \binom{n}{j} \frac{g_{k+n+j}^*(-1)}{2^j} \right) \right) \frac{x^k}{k!} \quad (99)$$

$$= \frac{(-1)^n}{n!2^n} \sum_{j=0}^n \binom{n}{j} \frac{1}{2^j} \sum_{k=0}^\infty \left(\frac{g_{k+n+j}^*(-1)}{k!} \right) x^k \quad (100)$$

$$= \frac{(-1)^n}{n!2^n} \sum_{j=0}^n \binom{n}{j} \frac{1}{2^j} D^{n+j} f(x) \quad (101)$$

$$= \frac{(-1)^n}{n!4^n} D^n \left(\sum_{j=0}^n \binom{n}{j} D^j 2^{n-j} \right) f(x) \quad (102)$$

$$= \frac{(-1)^n}{n!4^n} D^n (2 + D)^n f(x) \quad (103)$$

$$= \frac{(-1)^n}{n!4^n} D^n e^{-2x} D^n e^{2x} f(x). \quad (104)$$

Hence, if $f(x) \in \mathcal{L}-\mathcal{P}^s$, then $h_n(x) \in \mathcal{L}-\mathcal{P}^s$ and if $e^{2x}f(x) \in \mathcal{L}-\mathcal{P}^a$, then $e^{2x}h_n(x) \in \mathcal{L}-\mathcal{P}^a$ (see also Remark 8). \square

Equation (104) yields a little more information than Theorem 32, in particular we derive the recursive formula,

$$h_n(x) = \frac{-1}{4n} D e^{-2x} D e^{2x} h_{n-1}(x), \quad (n \geq 1, h_0(x) := f(x)). \quad (105)$$

Hence, only T_n needs to be diagonalizable with a Hermite multiplier sequence to establish that T_{n+1} is also diagonalizable with a Hermite multiplier sequence.

Given a Hermite diagonal differential operator, $T[H_n(x)] = \gamma_n H_n(x)$, $\gamma_n \in \mathbb{R}$, (see Definition 3), then T_0 (see 44) diagonalizes with the same eigenvalue sequence, namely $T_0[x^n] = \gamma_n x^n$. In fact, this is more generally known (see [1]). This indicates that if one assumes each operator T_n yields a Hermite multiplier sequence, then Theorem 32 has a trivial converse, in the sense that if one assumes each T_n diagonalizes with a Hermite multiplier sequence then T itself will also be hyperbolicity preserving. However, what if one only assumes that each T_n is hyperbolicity preserving? Must T be hyperbolicity preserving? We answer this question in the negative, with the following examples.

Example 33. Consider the following Hermite diagonal operator that is not hyperbolicity preserving (see Theorem 11),

$$T[H_n(x)] := ((-1)^{n+1}(n-1)) H_n(x). \quad (106)$$

Thus we calculate,

$$w(x) := \sum_{k=0}^{\infty} \frac{(-1)^{k+1}(k-1)}{k!} x^k = (x+1)e^{-x}. \quad (107)$$

Hence, using equation (104) (note, $f(x) = e^{-x}w(x)$ (see Theorem 11)), we can calculate the h_n 's,

$$h_0(x) := \sum_{k=0}^{\infty} \frac{Q_k^{(k)}(0)}{k!} x^k = (x+1)e^{-2x}, \quad (108)$$

$$h_1(x) := \sum_{k=0}^{\infty} \frac{Q_{k+2}^{(k)}(0)}{k!} x^k = \frac{1}{2}e^{-2x}, \quad \text{and} \quad (109)$$

$$h_n(x) := \sum_{k=0}^{\infty} \frac{Q_{k+2n}^{(k)}(0)}{k!} x^k = 0, \quad \text{for } n \geq 2. \quad (110)$$

Hence,

$$T = 1 - xD + \sum_{k=0}^{\infty} \left(\frac{\overbrace{h_0^{(k+2)}(0)}^{h_0^{(k+2)}(0)}}{(k+2)!} x^{k+2} + \frac{\overbrace{h_1^{(k)}(0)}^{h_1^{(k)}(0)}}{k!} x^k \right) D^{k+2} \quad (111)$$

$$= T_0 + T_2 D^2. \quad (112)$$

Thus,

$$T_0[x^n] = \left(1 - xD + \sum_{k=0}^{\infty} \frac{k(-2)^{k+1}}{(k+2)!} x^{k+2} D^{k+2} \right) x^n = ((-1)^{n+1}(n-1)) x^n, \quad (113)$$

$$T_2[x^n] = \left(\sum_{k=0}^{\infty} \left(\frac{-(-2)^{k-1}}{k!} x^k \right) D^k \right) x^n = \left(\frac{1}{2}(-1)^n \right) x^n, \quad \text{and} \quad (114)$$

$$T_{2m}[x^n] = (0) x^n = (0) x^n, \quad \text{for } m \geq 2. \quad (115)$$

We see that for every $n \geq 1$, $h_n(x) \in \mathcal{L}\text{-}\mathcal{P}^a$, hence T_{2n} is hyperbolicity preserving (see Theorem 10). However, the original operator T itself is not hyperbolicity preserving, as the following calculation shows,

$$T[4x^2 + 2x - 5] = T\left[\underbrace{-3H_0(x)}_{(-3)} + \underbrace{H_1(x)}_{(2x)} + \underbrace{H_2(x)}_{(4x^2 - 2)} \right] \quad (116)$$

$$= 1(-3) + 0(2x) + (-1)(4x^2 - 2) = -4x^2 - 1. \quad (117)$$

Example 34. Consider another Hermite diagonal operator that does not preserve hyperbolicity (see Theorem 11), $\{\gamma_k\}_{k=0}^{\infty} = \{(1/2)^k\}_{k=0}^{\infty}$; that is,

$$T[H_n(x)] = \gamma_n H_n(x) := (1/2)^n H_n(x). \quad (118)$$

Using Theorem 17 we write $T = \sum_{n=0}^{\infty} T_n D^n$, where $T_n[x^m] = b_{n,m} x^m$. We rewrite formula (104) in terms of $b_{n,m}$'s and γ_n 's (see Theorem 16),

$$\sum_{k=0}^{\infty} \frac{b_{n,k}}{k!} x^k = \frac{(-1)^n}{n!4^n} e^x D^n e^{-2x} D^n e^x \sum_{k=0}^{\infty} \frac{\gamma_k}{k!} x^k. \quad (119)$$

Since $\sum_{k=0}^{\infty} \frac{\gamma_k}{k!} x^k = e^{x/2}$, then

$$\sum_{k=0}^{\infty} \frac{b_{n,k}}{k!} x^k = \frac{(-1)^n}{n!4^n} \left(-\frac{1}{2} \right)^n \left(\frac{3}{2} \right)^n e^{x/2}. \quad (120)$$

Thus $\sum_{k=0}^{\infty} \frac{b_{n,k}}{k!} x^k \in \mathcal{L}\text{-}\mathcal{P}^s$ for every $n \in \mathbb{N}_0$. Hence, T_{2n} is hyperbolicity preserving for every $n \in \mathbb{N}_0$ (see Theorem 10), however, as noted above, T is not hyperbolicity preserving (see Theorem 11).

Example 35. To demonstrate the usefulness of Theorem 32, consider the following example. How would one show that

$$\{a_m\}_{m=0}^{\infty} := \{m^{5/2}\}_{m=0}^{\infty} \quad (121)$$

is not a multiplier sequence? Sequence $\{a_m\}_{m=0}^{\infty}$ satisfies the Turán inequalities and is a positive, increasing sequence. Thus some well known methods do not work (see for example see [24, p. 341],

concerning the Turán inequalities). One could apply the sequence to $(1+x)^5$ to calculate to the fifth associated Jensen polynomial,

$$= (5)x + (56.56\dots)x^2 + (155.88\dots)x^3 + (160)x^4 + (55.90\dots)x^5 \quad (122)$$

and verify that this polynomial has non-real zeros, however this can prove to be quite tedious. Instead, we apply Theorem 32 and calculate as summarized in Figure 1. Hence, after a few simple *numerical*

$b_{0,n}$	=	0,	1,	...
$b_{1,n}$	=	-1.41,	-3.65,	...
$b_{2,n}$	=	0.646,	0.804,	...
$b_{3,n}$	=	-0.0238,	-0.020,	...
\vdots		\vdots	\vdots	\ddots

FIGURE 1. Table of Hermite diagonal differential operator eigenvalues.

calculations we arrive at the highlighted portions in Figure 1 and note that they are negative and increasing, so $\{b_{3,n}\}_{n=0}^\infty$ is not a Hermite multiplier sequence (see Theorem 11). Thus, the original sequence, $\{a_m\}_{m=0}^\infty$, is not a Hermite multiplier sequence. Consequently, since $\{a_m\}_{m=0}^\infty$ is an increasing sequence that is not a Hermite multiplier sequence, by Theorem 11, we conclude that $\{a_m\}_{m=0}^\infty$ cannot be a classical multiplier sequence.

Our next task is to present several relationships between the polynomial coefficients, the Q_k 's, and the eigenvalues, the γ_k 's, in a Hermite diagonal differential operator,

$$T[H_n(x)] := \left(\sum_{k=0}^{\infty} Q_k(x) D^k \right) H_n(x) = \gamma_n H_n(x). \quad (123)$$

In general, in a diagonal differential operator, the relationship between the Q_k 's and the γ_k 's is not well understood, particularly in the context of hyperbolicity preservation. In special cases direct formulas have been found (see for example (87)) (cf. Theorem 9 and [8, Proposition 216, p. 107]), but a general relation has not been derived that indicates the properties of the Q_k 's and the γ_k 's for arbitrary hyperbolicity preserving operators. Thus, whenever possible, it is beneficial to present formulas that highlight the nature of the Q_k 's in terms of the eigenvalues, the γ_k 's. Using calculation (88) and (89), in Theorem 36 we can provide another formula for the Q_k 's in a Hermite diagonal differential operator.

Theorem 36. *Let $\{\gamma_n\}_{n=0}^\infty$ be a sequence of real numbers and $\{Q_k(x)\}_{k=0}^\infty$ be a sequence of real polynomials, such that*

$$T[H_n(x)] := \left(\sum_{k=0}^{\infty} Q_k(x) D^k \right) H_n(x) = \gamma_n H_n(x), \quad n \in \mathbb{N}_0. \quad (124)$$

Then for each $m \in \mathbb{N}_0$,

$$Q_m(x) = \sum_{k=0}^{[m/2]} \frac{(-1)^k}{k! 2^k} \left(\sum_{j=0}^k \binom{k}{j} \frac{g_{m-k+j}^*(-1)}{2^j} \right) \frac{x^{m-2k}}{(m-2k)!}, \quad (125)$$

where $\{g_k^(x)\}_{k=0}^\infty$ are the associated reversed Jensen polynomials of $\{\gamma_n\}_{n=0}^\infty$.*

We also derive a complex formulation for the Q_k 's in a Hermite diagonal differential operator (Theorem 38). A heuristic argument of the proof of Theorem 38 follows easily by considering the generating function of the Hermite polynomials (see [33, p. 187]),

$$e^{2xt-t^2} = \sum_{n=0}^{\infty} \frac{H_n(x)}{n!} t^n. \quad (126)$$

We now calculate $T[e^{2xt-t^2}]$ in two ways,

$$T[e^{2xt-t^2}] = \left(\sum_{k=0}^{\infty} Q_k(x) D^k \right) e^{2xt-t^2} = e^{2xt-t^2} \sum_{k=0}^{\infty} Q_k(x) (2t)^k, \quad \text{and} \quad (127)$$

$$T[e^{2xt-t^2}] = T \left[\sum_{n=0}^{\infty} \frac{H_n(x)}{n!} t^n \right] = \sum_{n=0}^{\infty} \frac{\gamma_n H_n(x)}{n!} t^n. \quad (128)$$

Hence,

$$\sum_{k=0}^{\infty} Q_k(x) (2t)^k = e^{-2xt+t^2} \left(\sum_{n=0}^{\infty} \frac{\gamma_n H_n(x)}{n!} t^n \right). \quad (129)$$

Thus, performing a Cauchy product on the right hand side of (129) and comparing the coefficients of t^n on the right and left of (129), for each $n \in \mathbb{N}_0$, we have,

$$Q_n(x) 2^n = \frac{1}{n!} \sum_{k=0}^n \binom{n}{k} \left(\frac{d^{n-k}}{dt^{n-k}} e^{-2xt+t^2} \right) \Big|_{t=0} \left(\frac{d^k}{dt^k} \sum_{j=0}^{\infty} \frac{\gamma_j H_j(x)}{j!} t^j \right) \Big|_{t=0} \quad (130)$$

$$= \frac{1}{n!} \sum_{k=0}^n \binom{n}{k} \frac{d^{n-k}}{dt^{n-k}} \sum_{j=0}^{\infty} \frac{H_j(ix)}{j!} (it)^j \Big|_{t=0} \frac{d^k}{dt^k} \sum_{j=0}^{\infty} \frac{\gamma_j H_j(x)}{j!} t^j \Big|_{t=0} \quad (131)$$

$$= \frac{1}{n!} \sum_{k=0}^n \binom{n}{k} i^{n-k} H_{n-k}(ix) \gamma_k H_k(x). \quad (132)$$

Remark 37. We must be cautious with the argument above since $T[e^{2xt-t^2}]$ need not converge and hence is only calculated formally. However, even under formal assumptions there is no reason to assume that a differential representation of a linear operator will calculate the same formal series as the operator itself. That is, the calculation,

$$T[e^{2xt-t^2}] = e^{2xt-t^2} \sum_{k=0}^{\infty} Q_k(x) (2t)^k, \quad (133)$$

has not been rigorously established.

Theorem 38. Let $\{\gamma_n\}_{n=0}^{\infty}$ be a sequence of real numbers and $\{Q_k(x)\}_{k=0}^{\infty}$ be a sequence of real polynomials, such that

$$T[H_n(x)] := \left(\sum_{k=0}^{\infty} Q_k(x) D^k \right) H_n(x) = \gamma_n H_n(x), \quad n \in \mathbb{N}_0. \quad (134)$$

Then for each $n \in \mathbb{N}_0$,

$$Q_n(x) = \frac{1}{n! 2^n} \sum_{k=0}^n \binom{n}{k} \gamma_k i^{n-k} H_{n-k}(ix) H_k(x). \quad (135)$$

Proof. Define

$$\tilde{T} := \sum_{k=0}^{\infty} Q_k(x) D^k, \quad (136)$$

where we define $Q_k(x)$ from equation (135). In the spirit of T. Forgács and A. Piotrowski [20, Theorem 3.1], we need only to show that $\tilde{T}[H_n(x)] = \gamma_n H_n(x)$ for each $n \in \mathbb{N}_0$. We note that for $n, m \in \mathbb{N}_0$, $D^m H_n(x) = 2^m \binom{m}{n} n! H_{n-m}(x)$ [33, p. 188]. We also note that $\binom{n}{k} \binom{k}{j} = \binom{n}{j} \binom{n-j}{k-j}$ (see [35, p. 3]). Using the generating function of the Hermite polynomials, equation (126), we now calculate

$$\tilde{T}[H_n(x)] = \sum_{k=0}^n \left(\frac{1}{k! 2^k} \sum_{j=0}^k \binom{k}{j} \gamma_j i^{k-j} H_{k-j}(ix) H_j(x) \right) D^k H_n(x) \quad (137)$$

$$= \sum_{k=0}^n \left(\frac{1}{k! 2^k} \sum_{j=0}^k \binom{k}{j} \gamma_j i^{k-j} H_{k-j}(ix) H_j(x) \right) 2^k \binom{n}{k} k! H_{n-k}(x) \quad (138)$$

$$= \sum_{j=0}^n \gamma_j H_j(x) \sum_{k=j}^n \binom{n}{k} \binom{k}{j} i^{k-j} H_{k-j}(ix) H_{n-k}(x) \quad (139)$$

$$= \sum_{j=0}^n \gamma_j H_j(x) \sum_{k=0}^{n-j} \binom{n}{k+j} \binom{k+j}{j} i^k H_k(ix) H_{(n-j)-k}(x) \quad (140)$$

$$= \sum_{j=0}^n \gamma_j H_j(x) \sum_{k=0}^{n-j} \binom{n}{j} \binom{n-j}{k} i^k H_k(ix) H_{(n-j)-k}(x) \quad (141)$$

$$= \sum_{j=0}^n \binom{n}{j} \gamma_j H_j(x) \cdot \sum_{k=0}^{n-j} \binom{n-j}{k} \left(\frac{d^k}{dt^k} e^{-2xt+t^2} \Big|_{t=0} \right) \cdot \left(\frac{d^{(n-j)-k}}{dt^{(n-j)-k}} e^{2xt-t^2} \Big|_{t=0} \right) \quad (142)$$

$$= \sum_{j=0}^n \binom{n}{j} \gamma_j H_j(x) \frac{d^{n-j}}{dt^{n-j}} e^{-2xt+t^2} e^{2xt-t^2} \Big|_{t=0} \quad (143)$$

$$= \gamma_n H_n(x). \quad \square$$

We can also establish an interesting relationship between alternating Hermite diagonal differential operators and non-alternating Hermite diagonal differential operators. This will allow us to provide an alternate proof and a non-obvious extension of T. Forgács and A. Piotrowski [20, Theorem 3.7] (cf. Example 13).

Theorem 39. Let $\{\gamma_k\}_{k=0}^\infty$ be a sequence of real numbers. Define the Hermite diagonal differential operators,

$$T[H_n(x)] := \left(\sum_{k=0}^\infty Q_k(x) D^k \right) H_n(x) = \gamma_n H_n(x) \quad (144)$$

and

$$\tilde{T}[H_n(x)] := \left(\sum_{k=0}^\infty \tilde{Q}_k(x) D^k \right) H_n(x) = (-1)^n \gamma_n H_n(x). \quad (145)$$

Then for each $n \in \mathbb{N}_0$,

$$Q_n(x) = \frac{(-2)^n}{n!} \left(\sum_{k=0}^\infty \frac{\tilde{Q}_k(x)}{2^k} D^k \right) x^n \quad (146)$$

and

$$\tilde{Q}_n(x) = \frac{(-2)^n}{n!} \left(\sum_{k=0}^\infty \frac{Q_k(x)}{2^k} D^k \right) x^n. \quad (147)$$

Proof. In light of Remark 37 and Theorem 38, we may conclude that,

$$\sum_{k=0}^\infty Q_k(x) (2t)^k = e^{-2xt+t^2} \left(\sum_{n=0}^\infty \frac{\gamma_n H_n(x)}{n!} t^n \right) \quad (148)$$

and

$$\sum_{k=0}^\infty \tilde{Q}_k(x) (2t)^k = e^{-2xt+t^2} \left(\sum_{n=0}^\infty \frac{(-1)^n \gamma_n H_n(x)}{n!} t^n \right); \quad (149)$$

i.e., as formal power series in t , the coefficients are equal (see [26] or [36, p. 130]). Hence, after substitution of $t \rightarrow -t$, we have

$$e^{-4xt} \sum_{k=0}^\infty Q_k(x) (-2t)^k = e^{-4xt} \left(e^{2xt+t^2} \left(\sum_{n=0}^\infty \frac{(-1)^n \gamma_n H_n(x)}{n!} t^n \right) \right) \quad (150)$$

$$= e^{-2xt+t^2} \sum_{n=0}^\infty \frac{(-1)^n \gamma_n H_n(x)}{n!} t^n \quad (151)$$

$$= \sum_{k=0}^\infty \tilde{Q}_k(x) (2t)^k. \quad (152)$$

Thus,

$$\tilde{Q}_n(x) = \frac{1}{n! 2^n} \frac{d^n}{dt^n} e^{-4xt} \sum_{k=0}^\infty Q_k(x) (-2t)^k \Big|_{t=0} \quad (153)$$

$$= \frac{1}{n! 2^n} \sum_{k=0}^n \binom{n}{k} (-4x)^{n-k} (-2)^k k! Q_k(x) \quad (154)$$

$$= \frac{(-2)^n}{n!} \sum_{k=0}^n \binom{n}{k} \frac{x^{n-k}}{2^k} k! Q_k(x) \quad (155)$$

$$= \frac{(-2)^n}{n!} \left(\sum_{k=0}^n \frac{Q_k(x)}{2^k} D^k \right) x^n. \quad (156)$$

By symmetry, equation (146) also holds. \square

Theorem 40 ([20, Theorem 3.7]). *Let $\{\gamma_k\}_{k=0}^\infty$ be a non-trivial Hermite multiplier sequence,*

$$T[H_n(x)] := \left(\sum_{k=0}^\infty Q_k(x) D^k \right) H_n(x) = \gamma_n H_n(x). \quad (157)$$

Then each $Q_k(x)$ has only real zeros.

Proof. If $\{\gamma_k\}_{k=0}^\infty$ is a Hermite multiplier sequence, then $\{(-1)^k \gamma_k\}_{k=0}^\infty$ is also a Hermite multiplier sequence [29, Proposition 119, p. 98]. Hence,

$$\sum_{k=0}^\infty \tilde{Q}_k(x) D^k, \quad (158)$$

is a hyperbolicity preserver. Thus, using the Borcea-Brändén Theorem [5, Theorem 5] (which requires non-trivial), we conclude that the operator,

$$\sum_{k=0}^\infty \frac{\tilde{Q}_k(x)}{2^k} D^k, \quad (159)$$

is also a hyperbolicity preserver. In particular, by Theorem 39, for each $n \in \mathbb{N}_0$,

$$Q_n(x) = \frac{(-2)^n}{n!} \left(\sum_{k=0}^\infty \frac{\tilde{Q}_k(x)}{2^k} D^k \right) x^n, \quad (160)$$

has only real zeros. \square

Theorem 39 actually shows that for every $k \in \mathbb{N}_0$, $Q_k(x)$ and $Q_{k+1}(x)$ have real interlacing zeros [8, Remark 6, p. 5]; i.e., for every $\alpha, \beta \in \mathbb{R}$, $k \in \mathbb{N}_0$, $\alpha Q_k(x) + \beta Q_{k+1}(x)$ has only real zeros.

We also note that Theorem 39 seems to indicate that only the polynomials $\{x^n\}_{n=0}^\infty$ are needed to establish that a Hermite diagonal differential operator is a hyperbolicity preserver. This observation provides us a new algebraic characterization of Hermite multiplier sequences (cf. [29, Theorem 46, p. 44]).

Theorem 41. *Let $\{\gamma_n\}_{n=0}^\infty$ be a non-zero, positive, classical multiplier sequence of real numbers and let T be a Hermite diagonal differential operator, where $T[H_n(x)] := \gamma_n H_n(x)$ for every $n \in \mathbb{N}_0$. Then T is hyperbolicity preserving if and only if,*

$$T[x^n] \in \mathcal{L}\text{-}\mathcal{P}, \quad (161)$$

for every $n \in \mathbb{N}_0$.

Proof. In order to establish the non-trivial direction, it suffices to show T is hyperbolicity preserving; i.e., $\{\gamma_n\}_{n=0}^\infty$ is a Hermite multiplier sequence. We will make use of the fact that $H'_n(x) = 2nH_{n-1}(x)$ for every $n \in \mathbb{N}$ [33, p. 188]. By assumption, for each $n \geq 2$, the following polynomial has only real zeros (see [33, p. 194] for the Hermite expansion of x^n),

$$D^{n-2}T[x^n] = D^{n-2}T \left[\frac{n!}{2^n} \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{1}{k!(n-2k)!} H_{n-2k}(x) \right] \quad (162)$$

$$= D^{n-2} \frac{n!}{2^n} \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{\gamma_{n-2k}}{k!(n-2k)!} H_{n-2k}(x) \quad (163)$$

$$= \frac{n!}{2^n} \left(\gamma_n \frac{2^{n-2}}{2!} H_2(x) + \gamma_{n-2} \frac{2^{n-2}}{1!} H_0(x) \right) \quad (164)$$

$$= n! \left(\frac{\gamma_n}{8} (4x^2 - 2) + \frac{\gamma_{n-2}}{4} (1) \right) \quad (165)$$

$$= \frac{n! \gamma_n}{4} \left(2x^2 + \left(\frac{\gamma_{n-2}}{\gamma_n} - 1 \right) \right). \quad (166)$$

Hence, $\frac{\gamma_{n-2}}{\gamma_n} \leq 1$ for every $n \geq 2$. Following the outline of A. Piotrowski [29, Theorem 127, p. 107], since $\{\gamma_n\}_{n=0}^\infty$ is assumed to be a multiplier sequence, then the Turán inequalities hold, $\gamma_{n-1}^2 - \gamma_{n-2}\gamma_n \geq 0$ for every $n \geq 2$. Hence, for each $n \geq 2$,

$$1 \leq \frac{\gamma_n}{\gamma_{n-2}} \leq \left(\frac{\gamma_{n-1}}{\gamma_{n-2}} \right)^2. \quad (167)$$

Thus, $\gamma_{n-2} \leq \gamma_{n-1}$ for $n \geq 2$, and therefore $\{\gamma_n\}_{n=0}^\infty$ is a Hermite multiplier sequence (see Theorem 7 and 11). \square

4. OPERATOR DIAGONALIZATIONS OF LAGUERRE MULTIPLIER SEQUENCES

The main objective of this section is exactly the same as that of the previous. We provide a few preliminary remarks for Laguerre multiplier sequences, we then find a formula for the $b_{n,k}$'s (see (43)), and finally we show that the $b_{n,k}$'s (see (43)) that arise from a Laguerre multiplier sequence yield more Laguerre multiplier sequences. The subtlety of the proceeding results can be seen in Examples 24-29, particularly Example 27.

Lemma 42. For $k, n \in \mathbb{N}_0$, the k^{th} derivative of the n^{th} Laguerre polynomial (Definition 1) evaluated at zero is,

$$L_n^{(k)}(0) = \binom{n}{k} (-1)^k. \quad (168)$$

Lemma 43. Let n, m , and p be integers. We then have the following combinatorial identity,

$$\begin{aligned} \sum_{k=0}^n \sum_{j=0}^m \binom{m}{j} \binom{k-j}{p-j} \binom{p}{k-j} \binom{n+1}{k+m-j} \\ = \binom{n+1}{p} \binom{n+1}{m} - \binom{n+1-m}{p-m} \binom{p}{n+1-m}. \end{aligned} \quad (169)$$

Proof. We first note that $\binom{n+1-m}{p-m} \binom{p}{n+1-m}$ can be added to the summation, hence, we wish to show,

$$\sum_{k=0}^{n+1} \sum_{j=0}^m \binom{m}{j} \binom{k-j}{p-j} \binom{p}{k-j} \binom{n+1}{k+m-j} = \binom{n+1}{p} \binom{n+1}{m}. \quad (170)$$

We perform a substitution of $l = k - j$ on the left side of (170) and then apply two Vandermonde identities [35, pp. 9, 15],

$$\sum_{k=0}^{n+1} \sum_{j=0}^m \binom{m}{j} \binom{k-j}{p-j} \binom{p}{k-j} \binom{n+1}{k+m-j} = \sum_{l=0}^{n+1} \binom{p}{l} \binom{n+1}{m+l} \sum_{j=0}^m \binom{m}{j} \binom{l}{p-j} \quad (171)$$

$$= \sum_{l=0}^{n+1} \binom{p}{l} \binom{n+1}{m+l} \binom{m+l}{p} \quad (172)$$

$$= \sum_{j=0}^{n+1} \binom{p}{j-m} \binom{n+1}{j} \binom{j}{p} \quad (173)$$

$$= \sum_{j=0}^{n+1} \binom{j}{p} \binom{p}{j-m} \binom{n+1}{j} \quad (174)$$

$$= \binom{n+1}{p} \binom{n+1}{m}. \quad \square$$

Theorem 44. Let T be a Laguerre diagonal differential operator, $T[L_n(x)] := \gamma_n L_n(x)$, where $\{\gamma_n\}_{n=0}^\infty$ a sequence of real numbers. Then there is a sequence of polynomials, $\{Q_k(x)\}_{k=0}^\infty$, and a sequence of classical diagonal differential operators, $\{T_n\}_{n=0}^\infty$, such that

$$T[L_n(x)] := \left(\sum_{k=0}^\infty Q_k(x) D^k \right) L_n(x) = \left(\sum_{k=0}^\infty T_k D^k \right) L_n(x) = \gamma_n L_n(x).$$

Then, for each $n \in \mathbb{N}_0$,

$$\{b_{n,m}\}_{m=0}^\infty := \left\{ \sum_{k=0}^m \binom{m}{k} \frac{(-1)^n}{n!} \left(\sum_{j=0}^n \binom{n}{j} \frac{(k+j)!}{((k+j)-n)!} g_{k+j}^*(-1) \right) \right\}_{m=0}^\infty,$$

where $T_n[x^m] = b_{n,m}x^m$ for every $n, m \in \mathbb{N}_0$.

Proof. The existence of the sequences $\{Q_k(x)\}_{k=0}^\infty$ and $\{T_k\}_{k=0}^\infty$ are established by Theorem 9 and 17. Recall from Theorem 17 that,

$$\{b_{n,m}\}_{m=0}^\infty = \left\{ \sum_{k=0}^m \binom{m}{k} Q_{k+n}^{(k)}(0) \right\}_{m=0}^\infty. \quad (175)$$

Hence, we wish to verify that,

$$Q_{k+n}^{(k)}(0) = \frac{(-1)^n}{n!} \left(\sum_{j=0}^n \binom{n}{j} \frac{(k+j)!}{((k+j)-n)!} g_{k+j}^*(-1) \right). \quad (176)$$

To ease the verification process, we first rewrite formula (176) as follows,

$$Q_n^{(m)}(0) = \sum_{p=0}^n (-1)^{n-m} \binom{n-m}{p-m} \binom{p}{n-m} g_p^*(-1). \quad (177)$$

We will now verify formula (177), *tour de force*, by induction. Suppose for every $m \in \mathbb{N}_0$ and $k \in \{0, 1, \dots, n\}$, formula (177) holds for $Q_k^{(m)}(0)$. We now calculate $Q_{n+1}^{(m)}(0)$ using the recursive formula of Theorem 9, equation (40), and Lemma 42 and 43,

$$Q_{n+1}^{(m)}(0) = \frac{1}{L_{n+1}^{(n+1)}} \left(\gamma_{n+1} L_{n+1}^{(m)}(0) - \sum_{k=0}^n \frac{d^m}{dx^m} [Q_k(x) L_{n+1}^{(k)}(x)] \Big|_{x=0} \right) \quad (178)$$

$$= (-1)^{n+1} \left(\sum_{p=0}^{n+1} \binom{n+1}{p} g_p^*(-1) \binom{n+1}{m} (-1)^m - \sum_{k=0}^n \sum_{j=0}^m \binom{m}{j} Q_k^{(j)}(0) L_{n+1}^{(k+m-j)}(0) \right) \quad (179)$$

$$= (-1)^{n+1} \left(\sum_{p=0}^{n+1} \binom{n+1}{p} g_p^*(-1) \binom{n+1}{m} (-1)^m - \sum_{k=0}^n \sum_{j=0}^m \binom{m}{j} \left(\sum_{p=0}^{n+1} \binom{k-j}{p-j} \binom{p}{k-j} (-1)^{k-j} g_p^*(-1) \right) \binom{n+1}{k+m-j} (-1)^{k+m-j} \right) \quad (180)$$

$$= \sum_{p=0}^{n+1} \left((-1)^{n+1-m} \binom{n+1}{p} \binom{n+1}{m} - \sum_{k=0}^n \sum_{j=0}^m \binom{m}{j} \binom{k-j}{p-j} \binom{p}{k-j} \binom{n+1}{k+m-j} \right) g_p^*(-1) \quad (181)$$

$$= \sum_{p=0}^{n+1} (-1)^{n+1-m} \binom{n+1-m}{p-m} \binom{p}{n+1-m} g_p^*(-1). \quad \square$$

Similar to the Hermite case (see Theorem 32) the following theorem establishes a Rodrigues type formula between $h_n(x)$ ($n \in \mathbb{N}_0$) and $f(x)$. This formula then relates the hyperbolicity preservation of T with each T_n ($n \in \mathbb{N}_0$).

Theorem 45. Suppose $\{\gamma_k\}_{k=0}^\infty$ is a non-trivial Laguerre multiplier sequence and let $\{g_k^*(x)\}_{k=0}^\infty$ be the reversed Jensen polynomials associated with $\{\gamma_k\}_{k=0}^\infty$. Then, for each $n \in \mathbb{N}_0$,

$$\{b_{n,m}\}_{m=0}^\infty := \left\{ \sum_{k=0}^m \binom{m}{k} \frac{(-1)^n}{n!} \left(\sum_{j=0}^n \binom{n}{j} \frac{(k+j)!}{((k+j)-n)!} g_{k+j}^*(-1) \right) \right\}_{m=0}^\infty,$$

is a Laguerre multiplier sequence.

Proof. By assumption, $\{\gamma_k\}_{k=0}^\infty$ is a Laguerre multiplier sequence. Hence, by Theorem 12,

$$f(x) = \sum_{k=0}^\infty \frac{f^{(k)}(0)}{k!} x^k := \sum_{k=0}^\infty g_k^*(-1) x^k \in \mathbb{R}[x] \cap \mathcal{L}\text{-}\mathcal{P}^s[-1, 0]. \quad (182)$$

To show that, $\{b_{n,m}\}_{m=0}^\infty$ is a Laguerre multiplier sequence we must show that,

$$h_n(x) := \sum_{m=0}^\infty \left(\sum_{k=0}^m \binom{m}{k} b_{n,k} (-1)^{m-k} \right) x^m \in \mathbb{R}[x] \cap \mathcal{L}\text{-}\mathcal{P}^s[-1, 0]. \quad (183)$$

We use Theorem 16 and perform the following calculations,

$$h_n(x) = \sum_{k=0}^\infty \left(\sum_{j=0}^k \binom{k}{j} b_{n,j} (-1)^{k-j} \right) x^k \quad (184)$$

$$= \sum_{k=0}^\infty \left(\frac{(-1)^n}{n!} \left(\sum_{j=0}^n \binom{n}{j} \frac{(k+j)!}{((k+j)-n)!} g_{k+j}^*(-1) \right) \right) x^k \quad (185)$$

$$= \frac{(-1)^n}{n!} \sum_{j=0}^n \binom{n}{j} \sum_{k=0}^\infty \left(\frac{(k+j)!}{((k+j)-n)!} g_{k+j}^*(-1) \right) x^k \quad (186)$$

$$= \frac{(-1)^n}{n!} \sum_{j=0}^n \binom{n}{j} \sum_{k=0}^\infty \frac{f^{(k+j)}(0)}{((k+j)-n)!} x^k \quad (187)$$

$$= \frac{(-1)^n}{n!} \sum_{j=0}^n \binom{n}{j} x^{n-j} D^n f(x) \quad (188)$$

$$= \frac{(-1)^n}{n!} (1+x)^n D^n f(x). \quad (189)$$

Hence, if $f(x) \in \mathbb{R}[x] \cap \mathcal{L}\text{-}\mathcal{P}^s[-1, 0]$, then $h_n(x) \in \mathbb{R}[x] \cap \mathcal{L}\text{-}\mathcal{P}^s[-1, 0]$. \square

Similar to the Hermite case, equation (189) also provides a recursive formula,

$$h_n(x) := \frac{-1}{n} (x+1)^n D(x+1)^{1-n} h_{n-1}(x), \quad (n \geq 1, h_0(x) := f(x)). \quad (190)$$

Thus, again, the hyperbolicity preservation of T_n with a Laguerre multiplier sequence, is enough to establish that T_{n+1} is hyperbolicity preserving with a Laguerre multiplier sequence.

Example 46. We show, similar to Examples 33 and 34, that it is possible for T_n to be hyperbolicity preserving for every n and yet T fail to be hyperbolicity preserving. Consider the following non-Laguerre multiplier sequence (see (193) and Theorem 12),

$$\{a_n\}_{n=0}^\infty := \{2, 3, 4, 5, 6, \dots\}, \quad (191)$$

where

$$T[L_n(x)] := a_n L_n(x). \quad (192)$$

From Theorem 17, we obtain $T = \sum_{n=0}^\infty T_n D^n$, where $T_n[x^m] = b_{n,m} x^m$ (see (43)) are classical diagonal differential operators. We calculate $f(x)$ from equation (182),

$$f(x) := \sum_{k=0}^\infty g_k^*(-1) x^k = x + 2. \quad (193)$$

Hence by formula (189),

$$h_0(x) = \sum_{k=0}^\infty \left(\sum_{j=0}^k \binom{k}{j} b_{0,j} (-1)^{k-j} \right) x^k = x + 2,$$

$$h_1(x) = \sum_{k=0}^{\infty} \left(\sum_{j=0}^k \binom{k}{j} b_{1,j} (-1)^{k-j} \right) x^k = -x - 1, \quad \text{and}$$

$$h_n(x) = \sum_{k=0}^{\infty} \left(\sum_{j=0}^k \binom{k}{j} b_{n,j} (-1)^{k-j} \right) x^k = 0, \quad \text{for } n \geq 2.$$

We see that $h_0(x) \notin \mathbb{R}[x] \cap \mathcal{L}\text{-}\mathcal{P}^s[-1, 0]$, hence $\{b_{0,k}\}_{k=0}^{\infty}$ is not a Laguerre multiplier sequence (see Theorem 12). However, if we define a classical multiplier sequence, $W[x^m] := \frac{1}{m!}x^m$, then

$$\sum_{k=0}^{\infty} \frac{b_{n,k}}{k!} x^k = e^x W[h_n(x)] \in \mathcal{L}\text{-}\mathcal{P}^s. \quad (194)$$

Hence, $\{b_{0,k}\}_{k=0}^{\infty}$ is a classical multiplier sequence (see Theorem 10). In addition, $h_n(x) \in \mathbb{R}[x] \cap \mathcal{L}\text{-}\mathcal{P}^s[-1, 0]$ for $n \geq 1$. Thus, each T_n ($n \geq 0$) is hyperbolicity preserving (see Theorem 12), each T_n ($n \geq 1$) diagonalizes with a Laguerre multiplier sequence, but T itself is not a hyperbolicity preserver.

From the calculations of (177) we can also provide a formula for the Q_k 's in a Laguerre differential operator (cf. Theorem 36 and 38).

Theorem 47. *Let $\{\gamma_n\}_{n=0}^{\infty}$ be a sequence of real numbers and $\{Q_k(x)\}_{k=0}^{\infty}$ be a sequence of polynomials, such that,*

$$T[L_n(x)] := \left(\sum_{k=0}^{\infty} Q_k(x) D^k \right) L_n(x) = \gamma_n L_n(x). \quad (195)$$

Then for each $n \in \mathbb{N}_0$,

$$Q_n(x) = \sum_{k=0}^n \left(\sum_{p=0}^n (-1)^{n-k} \binom{n-k}{p-k} \binom{p}{n-k} g_p^*(-1) \right) x^k, \quad (196)$$

where $\{g_k^*(x)\}_{k=0}^{\infty}$ are the associated reversed Jensen polynomials of $\{\gamma_n\}_{n=0}^{\infty}$.

Similar to Theorem 38, we provide another formula for the Q_k 's in a Laguerre diagonal differential operator (cf. [8, Proposition 216, p. 107]).

Theorem 48. *Let $\{\gamma_n\}_{n=0}^{\infty}$ be a sequence of real numbers and $\{Q_k(x)\}_{k=0}^{\infty}$ be a sequence of polynomials, such that,*

$$T[L_n(x)] := \left(\sum_{k=0}^{\infty} Q_k(x) D^k \right) L_n(x) = \gamma_n L_n(x). \quad (197)$$

Then for each $n \in \mathbb{N}_0$,

$$Q_n(x) = \sum_{k=0}^n \frac{(-x)^k}{k!} \sum_{j=0}^{n-k} \binom{n-k}{j} (-1)^j \gamma_j L_j(x). \quad (198)$$

Proof. The proof is very similar to the proof of Theorem 38. Define

$$\tilde{T} := \sum_{n=0}^{\infty} Q_n(x) D^n, \quad (199)$$

where $Q_n(x)$ is defined from equation (198). We will establish the result by showing that $\tilde{T}[L_m(x)] = \gamma_m L_m(x)$ for every $m \in \mathbb{N}_0$. Define the evaluation operator,

$$W := \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^n D^n. \quad (200)$$

Note that $W[f(x)] = f(0)$ for every polynomial $f(x)$. Using Theorem 42 and formula $\binom{n}{k} \binom{k}{j} = \binom{n}{j} \binom{n-j}{k-j}$ (see [35, p. 3]), we now evaluate \tilde{T} at $L_m(x)$,

$$\tilde{T}[L_m(x)] = \sum_{n=0}^m \left(\sum_{k=0}^n \frac{(-x)^k}{k!} \sum_{j=0}^{n-k} \binom{n-k}{j} (-1)^j \gamma_j L_j(x) \right) L_m^{(n)}(x) \quad (201)$$

$$= \sum_{j=0}^m (-1)^j \gamma_j L_j(x) \sum_{k=0}^m \sum_{n=0}^m \binom{n-k}{j} \frac{(-x)^k}{k!} L_m^{(n)}(x) \quad (202)$$

$$= \sum_{j=0}^m (-1)^j \gamma_j L_j(x) \sum_{k=0}^m \sum_{n=0}^m \binom{k}{j} \frac{(-x)^{n-k}}{(n-k)!} L_m^{(n)}(x) \quad (203)$$

$$= \sum_{j=0}^m (-1)^j \gamma_j L_j(x) \sum_{k=0}^m \binom{k}{j} \sum_{n=0}^m \frac{(-x)^n}{n!} L_m^{(n+k)}(x) \quad (204)$$

$$= \sum_{j=0}^m (-1)^j \gamma_j L_j(x) \sum_{k=0}^m \binom{k}{j} W[L_m^{(k)}(x)] \quad (205)$$

$$= \sum_{j=0}^m (-1)^j \gamma_j L_j(x) \sum_{k=0}^m \binom{k}{j} L_m^{(k)}(0) \quad (206)$$

$$= \sum_{j=0}^m (-1)^j \gamma_j L_j(x) \sum_{k=0}^m \binom{k}{j} \binom{m}{k} (-1)^k \quad (207)$$

$$= \sum_{j=0}^m (-1)^j \gamma_j L_j(x) \sum_{k=0}^m \binom{m}{j} \binom{m-j}{k-j} (-1)^k \quad (208)$$

$$= \sum_{j=0}^m (-1)^j \gamma_j L_j(x) \binom{m}{j} (-1)^j \sum_{k=0}^m \left(\binom{m-j}{k} (-1)^k \right) \quad (209)$$

$$= \sum_{j=0}^m \binom{m}{j} \gamma_j L_j(x) \sum_{k=0}^m \binom{m-j}{k} (-1)^k = \gamma_m L_m(x) \quad (210)$$

□

5. OPEN PROBLEMS

Problem 1. A frequent query in the literature is to find properties of the Q_k 's such that

$$T := \sum_{k=0}^{\infty} Q_k D^k \quad (211)$$

is hyperbolicity preserving. We ask instead a parallel question; what are the properties needed for classical diagonal differential operators, T_n 's, to form hyperbolicity preservers, as in

$$T = \sum_{k=0}^{\infty} T_n D^n \quad ? \quad (212)$$

Problem 2. Do the shifted Laguerre polynomials, $\{L_n(x - \alpha)\}_{n=0}^{\infty}$, possess the same property found in Theorem 44 and Theorem 45? Generalized Laguerre? Generalized Hermite?

Problem 3. Find all hyperbolicity preservers that can be written as a sum of classical hyperbolicity preservers ($T = \sum_{k=0}^{\infty} T_k D^k$), as in Theorem 17 or 19.

Problem 4. Does there exist a hyperbolicity preserver of the form,

$$T := \sum_{k=-\infty}^{\infty} T_k D^k, \quad (213)$$

such that $T_k \neq 0$ for every $k \in \mathbb{N}$? Compare with the open problem on “increasing degree” of A. Piotrowski [29, Problem 197, p. 172].

Problem 5. From T. Forgács and A. Piotrowski [20] we are given an intriguing open problem. Namely, if

$$T[H_n(x)] := \left(\sum_{k=0}^{\infty} Q_k(x) D^k \right) H_n(x) = \gamma_n H_n(x) \quad (214)$$

is a Hermite diagonal differential operator where $\{\gamma_n\}_{n=0}^{\infty}$ is a classic multiplier sequence and each $Q_k(x)$ has only real zeros, then can we conclude that T is a hyperbolicity preserver (cf. Theorem 40)?

Using the formulations throughout this paper, we pose two ideas that might prove of use to this question. First, following the method of T. Forgács and A. Piotrowski we analysis the leading and second-leading coefficients of the Q_k 's (for the definition of h_n , see equation (97)),

$$\frac{d}{dx}h_0(x) = \sum_{k=0}^{\infty} \frac{Q_{k+1}^{(k+1)}(0)}{k!} x^k = e^{-x}(f'(x) - f(x)), \quad \text{and} \quad (215)$$

$$(-4) \int h_1(x) dx = (-4) \sum_{k=0}^{\infty} \frac{Q_{k+1}^{(k-1)}(0)}{k!} x^k = e^{-x}(f'(x) + f(x)), \quad (216)$$

where $f(x) = \sum_{k=0}^{\infty} \frac{\gamma_k}{k!} x^k$ and where we take $Q_1^{(-1)}(0) := f'(0) + f(0)$. Thus, in general we ask; if $\{\gamma_k\}_{k=0}^{\infty}$ is a non-increasing, positive, classic multiplier sequence, then can we conclude that $e^{-x}(f'(x) - f(x))$ and $e^{-x}(f'(x) + f(x))$ have at least one Taylor coefficient of opposite sign?

Second, according to the Borcea-Branden Theorem [5, Theorem 5], if

$$T := \sum_{k=0}^{\infty} Q_k(x) D^k, \quad \text{and}, \quad W := \sum_{k=0}^{\infty} \frac{Q_k(x)}{2^k} D^k, \quad (217)$$

then T is hyperbolicity preserving if and only if W is hyperbolicity preserving (see also the proof of Theorem 39). However, if T is also a Hermite diagonal differential operator, then only the hyperbolicity of $T[x^n]$ is needed to conclude that T is a hyperbolicity preserver (see Theorem 41). Can the same be said of W ? This “minimal set” ($\{x^n\}_{n=0}^{\infty}$) that allows the conclusion of hyperbolicity preservation is a commonly sought after attribute of differential operators. We ask, what relationship do the sets A and B have, where

$$A = \left\{ p(x) : p(x) = \left(\sum_{k=0}^{\infty} Q_k(x) D^k \right) f_n(x) \right\}, \quad \text{and} \quad (218)$$

$$B = \left\{ p(x) : p(x) = \left(\sum_{k=0}^{\infty} Q_k(x) \alpha^k D^k \right) f_n(x) \right\}, \quad (219)$$

given $\{f_n(x)\}_{n=0}^{\infty}$ is some sequence of polynomials and $\alpha > 0$? If B only has hyperbolic polynomials, then must A have only hyperbolic polynomials? What restrictions would allow this conditional to hold?

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